Hints and Notes for Homework 7

problems 5-1 thru 5-5

If these seem simple and straightforward, that’s because they are. Your answer to each problem should be in the form of a single line

\[ Q = \text{configuration space} \]

where the configuration space is some combination of \( \mathbb{R}^n \) and/or \( T^m \) (the \( m \)-torus).

If these seem confusing and complicated, it’s probably due to some confusion between the concepts of configuration space and workspace or between different notations for revolute spaces. Here’s a primer on configuration space:

- **Note 1**: configuration space and workspace are entirely different animals. For example, consider the illustrations of workspaces in Figure 1.17 on page 18 of the text. Robots (a) and (b) have identical configuration spaces \( (Q = T^2 \times \mathbb{R}^1) \) while the figures show that their workspaces have different shapes. Interestingly, the cartesian robot of (d) has configuration space \( Q = \mathbb{R}^3 \) which happens to be exactly the same as its workspace, also \( \mathbb{R}^3 \). However, imagine adding three more prismatic joints to the cartesian robot; this 6-jointed robot would have configuration space \( Q = \mathbb{R}^6 \), while its workspace would remain the same as before, \( \mathbb{R}^3 \).

- **Note 2**: There is some confusion when it comes to notation for revolute spaces. Note that \( T^1 \), the 1-torus, is the same as \( S^1 \), the unit circle, is the same as \( SO(2) \), the group of all rotations in the plane.

\[ T^1 = S^1 = SO(2) \]

Confusingly, \( T^2 \), the 2-torus or doughnut, is NOT the same as \( S^2 \), the unit sphere, and neither are the same as \( SO(3) \), the group of all 3D rotations. While it is true that

\[ T^1 \times T^1 = T^2 \]

it is NOT true that \( S^1 \times S^1 = S^2 \), NOR is it the case that \( SO(2) \times SO(2) = SO(3) \). The correct description of the configuration space of a robot with two revolute joints is \( T^2 \), NOT \( S^2 \) NOR \( SO(3) \). I suggest sticking with \( \mathbb{R}^n \) and \( T^m \) notations for this problem.

problem 5-6

The problem wording might be a little confusing. There are 6 links total in the arm; the spherical wrist is the last 3 links of the 6-link anthropomorphic arm. All joints are revolute.

problem 5-7

Straightforward. Set up the equations, apply the gradient.
problem 5-8 thru 5-9 clarification

The words “planar” and “polygon” are causing confusion, and understandably so. Technically, all polygons are planar (see definition on Mathworld). However, Hutchinson intends that “polygon” in 5-10 be treated as a non-planar structure.

There is some debate about whether the point p in problem 5-9 lies in the same plane as the polygon. I think the “plane” part of this problem statement is meant to differentiate the planar “polygon” in problem 5-9 from the non-planar “polygon” in problem 5-10. As such, I would like you to consider the case where the point p is not necessarily in the same plane as the polygon.

Here’s the summary:

- **problem 5-8**: Since three points define a plane, this problem can only be considered as planar.
- **problem 5-9**: The word “polygon” here should be read to mean a planar shape with straight edges and a single face. The point p, however, should not be assumed to lie in the same plane as the face of the polygon.
- **problem 5-10**: The word “polygon” here should be read to mean “polyhedron,” a shape with straight edges and multiple faces. Again, the point p should not be assumed to lie in the same plane as any of the faces of the “polygon.”

problem 5-8

You are computing the minimum distance from a point to a line segment, not a line. This gives rise to two possible cases, depending on whether the perpendicular from point p intersects the line segment. The following notation may be helpful:

Let $A$ denote the line segment passing through $a_1$ and $a_2$.
Let $P$ be the line passing through $p$ that is perpendicular to $A$.

There are two cases to consider:

- **case 1**: $P$ intersects $A$. In this case, you should report the perpendicular distance from $p$ to $A$. It will be helpful to compute the point $a_\perp$ where $P$ intersects $A$. Let

$$a_\perp = a_1 + t_\perp \cdot (a_2 - a_1)$$

for some $t_\perp \in \mathbb{R}$. You can compute $t_\perp$ by using the fact that the dot product of $A$ and $P$ is zero. Once you know $t_\perp$, you know where $a_\perp$ is. (There are other ways of finding $a_\perp$ than the method I have elaborated on here. You need not use my method, however YOUR SOLUTION MUST GIVE AN EXPLICIT EXPRESSION FOR $a_\perp$ TO RECEIVE FULL CREDIT) Once $a_\perp$ is found, the minimum distance between $p$ and $A$ is simply

$$\|a_\perp - p\| .$$

- **case 2**: $P$ does not intersect $A$. In this case, the minimum distance from $p$ to $A$ is the smaller of the distances from $p$ to $a_1$ and $p$ to $a_2$. That is, the minimum distance is

$$\min \{ \|p - a_1\|, \|p - a_2\| \} .$$
problem 5-9

This problem is complicated! First of all, do not assume that \( p \) lies in the same plane as the polygon. Proceed according to the following algorithm:

1. compute the distances from \( p \) to each of the vertices.
2. compute the perpendicular distances for \( p \) to each of the edges.
3. compute the perpendicular distance from \( p \) to the face of the polygon.

The minimum distance from \( p \) to the polygon will be the minimum value reported by Parts 1 through 3. Part 1 is straightforward. For Part 2, simply invoke the result of problem 5-8. For part 3, we must

- compute the normal vector to the planar face of the polygon. You may assume that at least three of the vertices are not collinear; call them \( v_1, v_2, v_3 \). Give a valid expression for the normal vector, \( n \). (Hint: use the cross-product).

- compute the equation of the plane itself. All points in the plane will satisfy an equation of the form \( ax + by + cz = d \), and the vector \([a, b, c]\) which is equivalent to the normal vector \( n \). Find the equation of the plane.

Now solve for the point of intersection \( \ell \) of the plane with the line defined by the normal \( n \) that passes through point \( p \).

Finally, we must check to see whether \( \ell \) is inside or outside the face of the polygon. To do this, we can draw lines \( L_i \) from \( \ell \) to each of the vertices \( a_i \) of the polygon. Now, select any line in the plane to use as a angular reference. If the sum of the angles made by the lines \( L_i \) with the reference line is an integer multiple of \( 2\pi \), the intersection point \( \ell \) lies inside the face. If the point is inside the face of the polygon, we record the distance from the point \( p \) to \( \ell \). (for this part of the problem—the deduction of whether the intersection lies inside the polygon—it is not necessary that you show all the math; I will be satisfied if you simply copy the text of this procedure)

problem 5-10

Assume the 3D polygon can be decomposed into \( m \) flat faces \( G_i \).

problem 5-11

\( \rho(\alpha_i(q)) \) may look scary and complicated, but it’s really harmless. Distance \( \rho(\alpha_i(q)) \) is a function of the location in space \( \alpha_i(q) \). Consider writing this as \( \rho(x) \), where \( x \) is a vector in three dimensions. Thus, we can treat the gradient \( \nabla \) as the partial derivative w.r.t. the vector \( x \). The result now follows from the chain rule.
problem 5-12

I strongly recommend drawing a picture! You may make the following assumptions:

1. the repulsive forces act only on the four vertices of the robot.
2. the robot has three degrees of freedom and thus three “joint variables”

\[ q = \{x, y, \theta\}. \]

The robot is able to translate \( \{x, y\} \) (think of this as two prismatic joints) and rotate \( \{\theta\} \) (think of this as a single revolute joint).

Choose a value for \( \rho_0 \) for both obstacles so that the regions of influence do not overlap. Use equations (5.5) and (5.6) to construct the repulsive potential field \( U_{rep} \) and artificial workspace forces \( F_{rep} \). The math involved isn’t difficult, just tedious. Be careful that you don’t make silly algebra errors. You will need to find the values of \( \rho(a_i(q)) \) and \( \nabla \rho(a_i(q)) \) for each of the four vertices of the robot. Sum the repulsive forces for all vertices to get the net workspace force \( F \).

To find the configuration space forces and torque (there should be one component of your answer for each of \( x, y, \theta \)), follow Example 5.6. Each vertex \( a_i \) can be considered with coordinates \( a_x, a_y \) w.r.t. vertex \( a_1 \).

problem 5-15

You don’t need to go overboard solving this problem. Just explain a method for taking random, uniformly distributed samples from the interval \([0, 1]\) and use them to construct random orientations in \( SO(n) \). Do not restrict your attention to \( SO(3) \).